

A NOTE ON FINITE REAL MULTIPLE ZETA VALUES

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ABSTRACT. We prove three theorems on finite real multiple zeta values: the symmetric formula, the sum formula and the height-one duality theorem. These are analogues of their counterparts on finite multiple zeta values.

1. MAIN THEOREMS

For positive integers k_1, k_2, \dots, k_n with $k_1 \geq 2$, the multiple zeta value and the multiple zeta star value (MZV and MZSV, for short) are defined by

$$\zeta(k_1, k_2, \dots, k_n) := \sum_{m_1 > m_2 > \dots > m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}},$$

$$\zeta^*(k_1, k_2, \dots, k_n) := \sum_{m_1 \geq m_2 \geq \dots \geq m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}.$$

The finite real multiple zeta values (or symmetric multiple zeta values), which were first introduced by Kaneko and Zagier [10], are defined for any positive integers k_1, k_2, \dots, k_n as follows:

$$\zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n) := \sum_{i=0}^n (-1)^{k_1+k_2+\dots+k_i} \zeta^*(k_i, k_{i-1}, \dots, k_1) \zeta^*(k_{i+1}, k_{i+2}, \dots, k_n),$$

$$\zeta_{\mathcal{F}}^{\text{III}}(k_1, k_2, \dots, k_n) := \sum_{i=0}^n (-1)^{k_1+k_2+\dots+k_i} \zeta^{\text{III}}(k_i, k_{i-1}, \dots, k_1) \zeta^{\text{III}}(k_{i+1}, k_{i+2}, \dots, k_n).$$

Here, the symbols ζ^* and ζ^{III} on the right-hand sides stand for the regularized values coming from harmonic and shuffle regularizations respectively, i.e., real values obtained by taking constant terms of harmonic and shuffle regularizations as explained in [7]. In the sums, we understand $\zeta^*(\emptyset) = \zeta^{\text{III}}(\emptyset) = 1$.

Let \mathcal{Z} be the \mathbf{Q} -vector subspace of \mathbf{R} spanned by the MZVs. It is known that this is a \mathbf{Q} -algebra. In [10], Kaneko and Zagier proved that the difference $\zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n) - \zeta_{\mathcal{F}}^{\text{III}}(k_1, k_2, \dots, k_n)$ is in the principal ideal of \mathcal{Z} generated by $\zeta(2)$ (or π^2), in other words, that the congruence

$$\zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n) \equiv \zeta_{\mathcal{F}}^{\text{III}}(k_1, k_2, \dots, k_n) \pmod{\zeta(2)}$$

holds in \mathcal{Z} . They then defined the finite real multiple zeta value (FRMZV) $\zeta_{\mathcal{F}}(k_1, k_2, \dots, k_n)$ as an element in the quotient ring $\mathcal{Z}/\zeta(2)$ by

$$\zeta_{\mathcal{F}}(k_1, k_2, \dots, k_n) := \zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n) \pmod{\zeta(2)}.$$

We also refer to the values $\zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n)$ and $\zeta_{\mathcal{F}}^{\text{III}}(k_1, k_2, \dots, k_n)$ as (harmonic and shuffle versions of) finite real multiple zeta values.

In this paper, we prove the following theorems:

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Theorem 1.1 (Symmetric formula). *Let (k_1, k_2, \dots, k_n) be any index set ($k_i \in \mathbf{N}$) and let S_n be the symmetric group of degree n . Then, we have*

$$\sum_{\sigma \in S_n} \zeta_{\mathcal{F}}(k_{\sigma(1)}, k_{\sigma(2)}, \dots, k_{\sigma(n)}) = 0 \quad (\text{in } \mathcal{Z}/\zeta(2)).$$

Theorem 1.2 (Sum formula). *Let (k_1, k_2, \dots, k_n) be any index set ($k_i \in \mathbf{N}$). For positive integers k, n and i with $1 \leq i \leq n \leq k-1$, we have*

$$\sum_{\substack{k_1+k_2+\dots+k_n=k \\ k_i \geq 2}} \zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n) \equiv (-1)^{i-1} \left(\binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} \right) \zeta(k),$$

where the congruences are mod $\zeta(2)$ in the \mathbf{Q} -algebra \mathcal{Z} .

Theorem 1.3 (Height-one duality theorem). *For positive integers k and n , we have the equality*

$$\zeta_{\mathcal{F}}(k, \underbrace{1, \dots, 1}_{n-1}) = \zeta_{\mathcal{F}}(n, \underbrace{1, \dots, 1}_{k-1})$$

in $\mathcal{Z}/\zeta(2)$.

2. PROOFS

2.1. Proof of Theorem 1.1. Let $\mathbf{k}_1, \mathbf{k}_2$ and \mathbf{k} be any index sets. We note that the FRMZVs $\zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n)$ satisfy the harmonic product rule:

$$\zeta_{\mathcal{F}}^*(\mathbf{k}_1) \zeta_{\mathcal{F}}^*(\mathbf{k}_2) = \zeta_{\mathcal{F}}^*(\mathbf{k}_1 * \mathbf{k}_2),$$

where the right-hand-side is a linear combination of $\zeta_{\mathcal{F}}^*(\mathbf{k})$'s coming from the harmonic product in [5], e.g., $\zeta_{\mathcal{F}}^*((2) * (2)) = 2\zeta_{\mathcal{F}}^*(2, 2) + \zeta_{\mathcal{F}}^*(4)$.

Hoffman's theorem [6, Theorem 4.1] states that any symmetric sum

$$\sum_{\sigma \in S_n} \zeta_{\mathcal{F}}^*(k_{\sigma(1)}, k_{\sigma(2)}, \dots, k_{\sigma(n)})$$

is a polynomial in the Riemann zeta values $\zeta(k)$. His proof only uses the harmonic product rule of MZVs, and hence applies to our $\zeta_{\mathcal{F}}^*(\mathbf{k})$'s. Therefore, we conclude completely in the similar manner as in [6] that the symmetric sum above is a sum of products of $\zeta_{\mathcal{F}}^*(k) = (1 + (-1)^k)\zeta(k)$, which is 0 when k is odd and a multiple of $\zeta(2)$ when k is even.

Remark. One can also prove Theorem 1.1 directly by using the definition. For example, we compute

$$\begin{aligned} & \sum_{\sigma \in S_3} \zeta_{\mathcal{F}}^*(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}) \\ &= (1 + (-1)^{k_1})(1 + (-1)^{k_2})(1 + (-1)^{k_3}) \sum_{\sigma \in S_3} \zeta^*(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}) \\ &+ ((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})(\zeta(k_1 + k_2, k_3) + \zeta^*(k_3, k_1 + k_2)) \\ &+ ((-1)^{k_1} + (-1)^{k_3})(1 + (-1)^{k_2})(\zeta(k_1 + k_3, k_2) + \zeta^*(k_2, k_1 + k_3)) \\ &+ ((-1)^{k_2} + (-1)^{k_3})(1 + (-1)^{k_1})(\zeta(k_2 + k_3, k_1) + \zeta^*(k_1, k_2 + k_3)). \end{aligned}$$

When the weight (= sum of the indices) k is odd, the coefficients $(1 + (-1)^{k_1})(1 + (-1)^{k_2})(1 + (-1)^{k_3})$ and $((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})$ and etc. become 0. When k is even, the factor $(1 + (-1)^{k_1})(1 + (-1)^{k_2})(1 + (-1)^{k_3})$ becomes 0 if at least one of k_i 's are even, then $\sum_{\sigma \in S_3} \zeta^*(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}) = \zeta^*(k_1)\zeta^*(k_2)\zeta^*(k_3) - \zeta(k_1 + k_2)\zeta^*(k_3) - \zeta(k_1 +$

$k_3)\zeta^*(k_2) - \zeta(k_2 + k_3)\zeta^*(k_1) + 2\zeta(k_1 + k_2 + k_3)$ is 0 modulo $\zeta(2)$. As for the term $((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})(\zeta(k_1 + k_2, k_3) + \zeta^*(k_3, k_1 + k_2))$ etc., if we write this as $((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3})(\zeta(k_1 + k_2)\zeta^*(k_3) - \zeta(k_1 + k_2 + k_3))$, we see either $((-1)^{k_1} + (-1)^{k_2})(1 + (-1)^{k_3}) = 0$ or $(\zeta(k_1 + k_2)\zeta^*(k_3) - \zeta(k_1 + k_2 + k_3))$ is a multiple of $\zeta(2)$.

2.2. Proof of Theorem 1.2. We can prove Theorem 1.2 exactly in the same manner as in [12]. Set

$$S_{k,n,i} := \sum_{\substack{k_1+k_2+\dots+k_n=k \\ k_i \geq 2}} \zeta_{\mathcal{F}}^*(k_1, k_2, \dots, k_n).$$

We notice that the harmonic version of FRMZVs satisfy the harmonic product rule. Thus, $S_{k,n,i}$ enjoy the recursion relation in the following lemma, which can be proved exactly in the same way as in [12, Proposition 2.2].

Lemma 2.1. *For positive integers k, n and i with $2 \leq i+1 \leq n \leq k-1$, we have*

$$(n-i)S_{k,n,i} + iS_{k,n,i+1} + (k-n)S_{k,n-1,i} = 0.$$

We prove Theorem 1.2 by backward induction on n . To this, we need the initial value.

Lemma 2.2. *For positive integers k and i with $1 \leq i \leq k-1$, we have*

$$S_{k,k-1,i} \equiv (-1)^{i-1} \binom{k}{i} \zeta(k) \pmod{\zeta(2)}.$$

Proof. Since $S_{k,k-1,i} = \zeta_{\mathcal{F}}^*(\underbrace{1, \dots, 1}_{i-1}, 2, \underbrace{1, \dots, 1}_{k-i-1})$, we compute $\zeta_{\mathcal{F}}^{\text{III}}(\underbrace{1, \dots, 1}_{i-1}, 2, \underbrace{1, \dots, 1}_{k-i-1})$ instead.

Because of the fact $\zeta_{\mathcal{F}}^{\text{III}}(1, \dots, 1) = 0$, we have by definition

$$\begin{aligned} S_{k,k-1,i} &\equiv \zeta_{\mathcal{F}}^{\text{III}}(\underbrace{1, \dots, 1}_{i-1}, 2, \underbrace{1, \dots, 1}_{k-i-1}) \pmod{\zeta(2)} \\ &= \zeta^{\text{III}}(\underbrace{1, \dots, 1}_{i-1}, 2, \underbrace{1, \dots, 1}_{k-i-1}) + (-1)^k \zeta^{\text{III}}(\underbrace{1, \dots, 1}_{k-i-1}, 2, \underbrace{1, \dots, 1}_{i-1}). \end{aligned}$$

By using [7, eq.(5.2)] for $w_0 = xy^l$, we have $\zeta^{\text{III}}(\underbrace{1, \dots, 1}_m, 2, \underbrace{1, \dots, 1}_{l-1}) = (-1)^m \binom{m+l}{m} \zeta(2, \underbrace{1, \dots, 1}_{m+l-1})$.

Thus,

$$S_{k,k-1,i} \equiv (-1)^{i-1} \left(\binom{k-1}{i-1} + \binom{k-1}{i} \right) \zeta(2, \underbrace{1, \dots, 1}_{k-2}) = (-1)^{i-1} \binom{k}{i} \zeta(k) \pmod{\zeta(2)}.$$

■

Let us consider the case $n = k-1$ of Theorem 1.2. If k is even, the identity holds from Lemma 2.2. If k is odd, then n is even and the identity again follows because

$$\binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} = \binom{k-1}{i-1} + \binom{k-1}{i} = \binom{k}{i}.$$

We assume the identity holds for n . By Lemma 2.1,

$$\begin{aligned}
(n-k)S_{k,n-1,i} &= (n-i)S_{k,n,i} + iS_{k,n,i+1} \\
&= (n-i)(-1)^{i-1} \left(\binom{k-1}{i-1} + (-1)^n \binom{k-1}{n-i} \right) \zeta(k) \\
&\quad + i(-1)^i \left(\binom{k-1}{i} + (-1)^n \binom{k-1}{n-i-1} \right) \zeta(k) \\
&= (-1)^{i-1} \left((n-i) \binom{k-1}{i-1} + (k-n+i)(-1)^n \binom{k-1}{n-i-1} \right) \zeta(k) \\
&\quad + (-1)^i \left((k-i) \binom{k-1}{i} + i(-1)^n \binom{k-1}{n-i-1} \right) \zeta(k) \\
&= (n-k)(-1)^{i-1} \left(\binom{k-1}{i-1} + (-1)^{n-1} \binom{k-1}{n-i-1} \right) \zeta(k).
\end{aligned}$$

Thus, the identity holds for $n-1$.

Remark. We mention an analogy of Theorem 1.2 on finite real multiple zeta star values. For positive integers k_1, k_2, \dots, k_n , let us define $\zeta_{\mathcal{F}}^{*,*}$ by

$$\zeta_{\mathcal{F}}^{*,*}(k_1, k_2, \dots, k_n) := \sum_{\substack{\circ \text{ is either a comma ``,'''} \\ \text{or a plus ``+''}}} \zeta_{\mathcal{F}}^*(k_1 \circ k_2 \circ \dots \circ k_n).$$

Set $S_{k,n,i}^* := \sum_{\substack{k_1+k_2+\dots+k_n=k \\ k_i \geq 2}} \zeta_{\mathcal{F}}^{*,*}(k_1, k_2, \dots, k_n)$. Since these $\zeta_{\mathcal{F}}^{*,*}(k_1, k_2, \dots, k_n)$ satisfy the same harmonic product rule as $\zeta^*(k_1, k_2, \dots, k_n)$, $S_{k,n,i}^*$ enjoy the same recursion relation as [12, Proposition 2.2], that is, $(n-i)S_{k,n,i}^* + iS_{k,n,i+1}^* - (k-n)S_{k,n-1,i}^* = 0$. Writing $\mathbf{k}_i \sqcup \mathbf{k}_j$ for juxtaposition of index sets \mathbf{k}_i and \mathbf{k}_j , we see from [6, Theorem 3.1] that

$$\zeta_{\mathcal{F}}^{*,*}(k_n, k_{n-1}, \dots, k_1) = (-1)^n \sum_{\mathbf{k}_1 \sqcup \dots \sqcup \mathbf{k}_l = (k_1, k_2, \dots, k_n)} (-1)^l \zeta_{\mathcal{F}}^*(\mathbf{k}_1) \cdots \zeta_{\mathcal{F}}^*(\mathbf{k}_l).$$

Consider the case $(k_1, k_2, \dots, k_n) = (\underbrace{1, \dots, 1}_{k-i-1}, 2, \underbrace{1, \dots, 1}_{i-1})$ in this equality. Since $\zeta_{\mathcal{F}}^*(1, \dots, 1) \equiv \zeta_{\mathcal{F}}^{\text{III}}(1, \dots, 1) = 0 \pmod{\zeta(2)}$, the right-hand-side is equal modulo $\zeta(2)$ to $\zeta_{\mathcal{F}}^{\text{III}}(\underbrace{1, \dots, 1}_{i-1}, 2, \underbrace{1, \dots, 1}_{k-i-1})$.

Thus, we find $S_{k,k-1,i}^* \equiv S_{k,k-1,i} \equiv (-1)^{i-1} \binom{k}{i} \zeta(k) \pmod{\zeta(2)}$. In a similar way as the proof of Theorem 1.2 (i.e., by backward induction on n), we get

$$\sum_{\substack{k_1+k_2+\dots+k_n=k \\ k_i \geq 2}} \zeta_{\mathcal{F}}^{*,*}(k_1, k_2, \dots, k_n) \equiv (-1)^{i-1} \left((-1)^n \binom{k-1}{i-1} + \binom{k-1}{n-i} \right) \zeta(k) \pmod{\zeta(2)}.$$

2.3. Proof of Theorem 1.3. For a given index \mathbf{k} , we call the number of its elements greater than 1 the height. With this terminology, we shall call

$$\zeta_{\mathcal{F}}(k, \underbrace{1, \dots, 1}_{n-1})$$

height one FRMZVs. In this section, we prove Theorem 1.3. To this, we state the following key lemma.

Lemma 2.3. *For positive integers k and n with $k \geq 2$, we have*

$$\zeta^{\text{III}}(\underbrace{1, \dots, 1}_{n-1}, k) = (-1)^{n-1} \zeta^{\star}(k, \underbrace{1, \dots, 1}_{n-1}).$$

Proof. We note that the MZVs $\zeta^{\text{III}}(k_1, k_2, \dots, k_n)$ satisfy the shuffle product rule (for precise definition in [5]) coming from the iterated integral expressions of the MZVs: $\zeta^{\text{III}}(\mathbf{k}_1)\zeta^{\text{III}}(\mathbf{k}_2) = \zeta^{\text{III}}(\mathbf{k}_1 \sqcup \mathbf{k}_2)$, e.g., $\zeta^{\text{III}}((1, 1) \sqcup (2)) = 3\zeta^{\text{III}}(2, 1, 1) + 2\zeta^{\text{III}}(1, 2, 1) + \zeta^{\text{III}}(1, 1, 2)$. Here, the notation $\mathbf{k}_1 \sqcup \mathbf{k}_2$ is a \mathbf{Z} -linear combination of indices and we extend ζ^{III} linearly. To make notations easier, let $\zeta^{\text{III}}(1 \oplus (k_1, k_2, \dots, k_n)) = \zeta^{\text{III}}(k_1 + 1, k_2, \dots, k_n)$ and $\zeta^{\text{III}}(\dots, l, \underbrace{1, \dots, 1}_{-1}, m, \dots) = \zeta^{\text{III}}(\dots, l + m - 1, \dots)$. By the regularization formula [7, eq.(5.2)], we have (extending $1 \oplus (\cdot)$ also linearly)

$$\begin{aligned} \zeta^{\text{III}}(\underbrace{1, \dots, 1}_{n-1}, k) &= (-1)^{n-1} \zeta^{\text{III}}(1 \oplus ((\underbrace{1, \dots, 1}_{n-1}) \sqcup (k-1))) \\ &= (-1)^{n-1} \sum_{\substack{a_1 + \dots + a_k = n-1 \\ a_i \geq 0 (i=1, 2, \dots, k)}} \zeta(2, \underbrace{1, \dots, 1}_{a_1-1}, \dots, 2, \underbrace{1, \dots, 1}_{a_{k-2}-1}, 2, \underbrace{1, \dots, 1}_{a_{k-1}+a_k}) \\ &= (-1)^{n-1} \sum_{\substack{a_1 + \dots + a_{k-1} = n-1 \\ a_i \geq 0 (i=1, 2, \dots, k-1)}} (a_{k-1} + 1) \zeta(2, \underbrace{1, \dots, 1}_{a_1-1}, \dots, 2, \underbrace{1, \dots, 1}_{a_{k-2}-1}, 2, \underbrace{1, \dots, 1}_{a_{k-1}}) \\ &= (-1)^{n-1} \sum_{\substack{a_1 + \dots + a_{k-1} = n-1 \\ a_i \geq 0 (i=1, 2, \dots, k-1)}} (a_{k-1} + 1) \zeta(a_{k-1} + 2, a_{k-2} + 1, \dots, a_1 + 1). \end{aligned}$$

For the last equality, we used the duality formula of MZVs. That the last sum equals $\zeta^{\star}(n, \underbrace{1, \dots, 1}_{k-1})$ is due to Ohno [11, Proof of Theorem 2], see also [9, §3]. Thus

$$\zeta^{\text{III}}(\underbrace{1, \dots, 1}_{n-1}, k) = (-1)^{n-1} \zeta^{\star}(k, \underbrace{1, \dots, 1}_{n-1}).$$

■

Now, we prove Theorem 1.3. When either k or $n = 1$, the theorem clearly holds. We consider the case when $k, n \geq 2$. From the above Lemma 2.3, we have

$$\begin{aligned} &\zeta_{\mathcal{F}}^{\text{III}}(k, \underbrace{1, \dots, 1}_{n-1}) - \zeta_{\mathcal{F}}^{\text{III}}(n, \underbrace{1, \dots, 1}_{k-1}) \\ &= \zeta(k, \underbrace{1, \dots, 1}_{n-1}) + (-1)^k \zeta^{\star}(k, \underbrace{1, \dots, 1}_{n-1}) - (\zeta(n, \underbrace{1, \dots, 1}_{k-1}) + (-1)^n \zeta^{\star}(n, \underbrace{1, \dots, 1}_{k-1})). \end{aligned}$$

Let $\psi(X) = \frac{\Gamma'(X)}{\Gamma(X)}$. By using the well-known generating series

$$\begin{aligned} 1 - \sum_{k, n \geq 1} \zeta(k+1, \underbrace{1, \dots, 1}_{n-1}) X^k Y^n &= \exp\left(\sum_{n \geq 2} \zeta(n) \frac{X^n + Y^n - (X+Y)^n}{n}\right) \\ &= \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} \end{aligned}$$

(cf. Aomoto [2] and Drinfel'd [3]) and $\psi(1-X) = -\sum_{k \geq 2} \zeta(k)X^{k-1} - \gamma$ (γ is Euler's constant.), we have

$$\begin{aligned} & \sum_{k,n \geq 2} \left(\zeta(k, \underbrace{1, \dots, 1}_{n-1}) - \zeta(n, \underbrace{1, \dots, 1}_{k-1}) \right) X^{k-1} Y^{n-1} \\ &= \left(\frac{1}{Y} - \frac{1}{X} \right) \left(1 - \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} \right) + \psi(1-X) - \psi(1-Y). \end{aligned}$$

On the other hand, from Kaneko and Ohno [8, Theorem 2],

$$\begin{aligned} & \sum_{k,n \geq 2} \left((-1)^k \zeta^\star(k, \underbrace{1, \dots, 1}_{n-1}) - (-1)^n \zeta^\star(n, \underbrace{1, \dots, 1}_{k-1}) \right) X^{k-1} Y^{n-1} \\ &= -\psi(X) + \psi(Y) - \pi(\cot(\pi X) - \cot(\pi Y)) \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}, \end{aligned}$$

From these, and by the well-known equalities:

$$\begin{aligned} \pi \cot(\pi X) &= \frac{1}{X} + \psi(1-X) - \psi(1+X), \\ \psi(X) &= \psi(1+X) - \frac{1}{X}, \end{aligned}$$

we have

$$\begin{aligned} & \sum_{k,n \geq 2} \left(\zeta_{\mathcal{F}}^{\text{III}}(k, \underbrace{1, \dots, 1}_{n-1}) - \zeta_{\mathcal{F}}^{\text{III}}(n, \underbrace{1, \dots, 1}_{k-1}) \right) X^{k-1} Y^{n-1} \\ &= \left(\frac{1}{Y} - \frac{1}{X} \right) \left(1 - \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} \right) + \psi(1-X) - \psi(1-Y) \\ &\quad - \psi(X) + \psi(Y) - \pi(\cot(\pi X) - \cot(\pi Y)) \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} \\ &= \left(1 - \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} \right) (\psi(1-X) - \psi(1+X) - \psi(1-Y) + \psi(1+Y)) \\ &= -2 \left(1 - \frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)} \right) \sum_{l \geq 1} \zeta(2l) (X^{2l-1} - Y^{2l-1}). \end{aligned}$$

Since the coefficients of $\frac{\Gamma(1-X)\Gamma(1-Y)}{\Gamma(1-X-Y)}$ belong to the \mathbf{Q} -algebra \mathcal{Z} , we have

$$\zeta_{\mathcal{F}}^{\text{III}}(k, \underbrace{1, \dots, 1}_{n-1}) \equiv \zeta_{\mathcal{F}}^{\text{III}}(n, \underbrace{1, \dots, 1}_{k-1}) \pmod{\zeta(2)}.$$

This proves Theorem 1.3.

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